The Reflection Principle For A Brownian Motion With Zero Drift

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The payoff on a plain vanilla option depends only on the final value of the underlying asset. Options whose payoffs depend on the path of the underlying asset are called path-dependent or exotic. A barrier option is an exotic that has an explicit pricing formula based on the reflection principle for Brownian motion. To price this option we need the joint density of a Brownian motion with drift and its maximum to date. In this section we will develop the mathematics of the reflection principle for a Brownian motion with zero drift.

We will define the variable W_T to be the value of a Brownian motion with zero drift at time T and the variable δW_t to be the change in the value of the Brownian motion at time t where $0 \leq t \leq T$. The equation for the value of the Brownian motion at time T is...

$$W_T = W_0 + \int\limits_0^T \delta W_t \text{ ...where... } W_0 = 0 \text{ ...and... } \delta W_t = W_t - W_{t-1} \text{ ...and... } \delta W_t \sim N\Big[0, \delta t\Big] \tag{1}$$

The Reflection Principle in Discrete Time

The scaled symmetric random walk in the table below starts at point A at time zero where it has a value equal to zero. At each time step the random walk goes up with probability 0.50 or goes down with probability 0.50. If it goes up the value of the random walk increases by one and if it goes down the value of the random walk decreases by one. The random walk has 9 time steps, 512 possible paths (2^9), a maximum value of 9 and a minimum value of -9. Our task is to count the paths that reach or exceed the barrier m before time $T = 9$ and end up at or below w at time $T = 9$.

Table 1 - The Scaled Symmetric Random Walk

T=0	T=1	T=2	T=3	T=4	T=5	T=6	T=7	T=8	T=9	#Paths	Value
									J1	1	9
								I1			
							H1		J2	9	7
2m-w	—	—	—	—	—	G1	—	I2	—		
					F1		H2		J3	36	5
m	—	—	—	E1	—	G2	—	I3	—		
			D1		F2		H3		J4	84	3
w	—	C1	—	E2	—	G3	—	I4	—		
	B1		D2		F3		H4		J5	126	1
A		C2		E3		G4		I5			
	B2		D3		F4		H5		J6	126	(1)
		C3		E4		G5		I6			
			D4		F5		H6		J7	84	(3)
				E5		G6		I7			
					F6		H7		J8	36	(5)
						G7		I8			
							H8		J9	9	(7)
								I9			
									J10	1	(9)

Per the random walk in the table above the points E1, G2 and I3 are on the barrier m. If once the random walk reaches the barrier we restart the random walk from any of these three points we will find that for every path that leads to a point below the barrier m there is a reflected path that leads to a point above the barrier m. Each of these path pairs is a reflection because whereas both paths starts at the same point when the first random walk increases at any given time step the second decreases at that time step and visa versa. This is the **reflection principle**. We can take this a step further and say that for the random walk that restarts at the barrier and leads to a point below w there is a reflected path that leads to a point above $2m - w$. These paths and their reflections are listed in the following table...

Table 2 - Reflected Paths of the Random Walk Restarted at the Barrier

Path Pair	Leads To Point Below w	Leads To Point Above $2m - w$
1 and 2	E1-F2-G3-H4-I5-J6	E1-F1-G1-H1-I1-J1
3 and 4	E1-F2-G3-H4-I5-J5	E1-F1-G1-H1-I1-J2
5 and 6	E1-F2-G3-H4-I4-J5	E1-F1-G1-H1-I2-J2
7 and 8	E1-F2-G3-H3-I4-J5	E1-F1-G1-H2-I2-J2
9 and 10	E1-F2-G2-H3-I4-J5	E1-F1-G2-H2-I2-J2
11 and 12	E1-F1-G2-H3-I4-J5	E1-F2-G2-H2-I2-J2
13 and 14	G2-H3-I4-J5	G2-H2-I2-J2

For a path to reach or exceed the barrier m before time $T = 9$ and end up at or below w at time $T = 9$ the following two conditions must be met: (1) the maximum point on the path must be greater than or equal to m and (2) the end point on the path must be less than or equal to w. The reflection principle is very useful because rather than count the paths that reach or exceed m and end up at or below w (two conditions) we can simply count the paths that end up at or above $2m - w$ (one condition) since the path count will be the same.

By using the reflection principle we can say that the number of paths that reach or exceed m before time $T = 9$ and end up at or below w at time $T = 9$ is equal to the number of paths that end up at or above $2m - w$ by time $T = 9$, which according to Table 1 above is 10 (9 paths lead to J2 and 1 path leads to J1). To prove to ourselves that 10 is indeed the right answer the paths that reach or exceed m before time $T = 9$ and end up at or below w at time $T = 9$ are...

Table 3 - Paths That Hit Barrier m And End Up At Or Below w By $T = 9$

Number	Path
1	A-B1-C1-D1-E1-F2-G3-H4-I5-J6
2	A-B1-C1-D1-E1-F2-G3-H4-I5-J5
3	A-B1-C1-D1-E1-F2-G3-H4-I4-J5
4	A-B1-C1-D1-E1-F2-G3-H3-I4-J5
5	A-B1-C1-D1-E1-F2-G3-H3-I4-J5
6	A-B1-C1-D1-E1-F1-G2-H3-I4-J5
7	A-B1-C1-D1-E2-F2-G2-H3-I4-J5
8	A-B1-C1-D2-E2-F2-G2-H3-I4-J5
9	A-B1-C2-D2-E2-F2-G2-H3-I4-J5
10	A-B2-C2-D2-E2-F2-G2-H3-I4-J5

Note that just as 9 paths lead to J2, which is two steps above the barrier, and 1 path leads to J1, which is three steps above the barrier, 9 reflected paths lead to J5, which is two steps below the barrier, and 1 reflected path leads to J6, which is three steps below the barrier.

The Reflection Principle in Continuous Time

Note that the distribution of the ending value of the Brownian motion in Equation (1) above at time T is...

$$W_T \sim N\Big[0, T\Big] \tag{2}$$

We will make the following definitions...

$$M_T^+ \;=\; \text{Maximum value of a Brownian motion during the time interval } [0,\, T]$$
$$M_T^- \;=\; \text{Minimum value of a Brownian motion during the time interval } [0,\, T]$$

Note that if the Brownian motion starts at zero at time zero then the maximum value of the Brownian motion (M_T^+) must be greater than or equal to zero and the minimum value of the Brownian motion (M_T^-) must be less than or equal to zero.

Important - If the analyst uses Equation (1) above to simulate the path of the Brownian motion then to accurately calculate the percent of times that the Brownian motion hits the barrier over the time interval $[0, T]$ then (1) the number of trials must be large and (2) the time interval $[0, T]$ must be broken down into very small time buckets such that running a Monte Carlo simulation may be very time consuming due to the extreme number of calculations.

Given that $m \geq 0$ and $w \leq m$ the equation for the probability that a Brownian motion with zero drift and variance v crosses the barrier m sometime during the time interval [0,T] and ends up below w at time T is...

$$Prob\Big[M_T^+ > m, W_T < w\Big] = Prob\Big[W_T > 2m - w\Big] = \int\limits_{2m-w}^{\infty} \frac{1}{\sqrt{2\pi v}}\, e^{-\frac{1}{2v}z^2}\,\delta z \tag{3}$$

Note that in continuous time the probability of a Brownian motion landing on a point exactly on the barrier m or landing exactly on point w is zero.

We will define a new random variate x that is the normalized random variate z in Equation (3) above...

$$x = \frac{z - \mu_z}{\sigma_z} = \frac{z}{\sqrt{v}} \;\; ...such\; that... \;\; \frac{\delta x}{\delta z} = \frac{1}{\sqrt{v}} \;\; ...and... \;\; \delta z = \delta x \sqrt{v} \tag{4}$$

After normalization the equation for the probability as defined by Equation (3) becomes...

$$\begin{aligned} P\Big[M_T > m, W_T < w\Big] &= \int\limits_{2m-w}^{\infty} \frac{1}{\sqrt{2\pi}\sqrt{v}}\, e^{-\frac{1}{2v}z^2}\,\delta z \\ &= \int\limits_{\frac{2m-w}{\sqrt{v}}}^{\infty} \frac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2}x^2}\,\delta x \\ &= 1 - N\left[\frac{2m - w}{\sqrt{v}}\right] \end{aligned} \tag{5}$$

Because...

$$z^2 = x^2 v \;\; ...and... \;\; \delta z = \delta x \sqrt{v} \tag{6}$$

Given that $m \leq 0$ and $w \geq m$ the equation for the probability that a Brownian motion with zero drift and variance v crosses the barrier m sometime during the time interval [0,T] and ends up above w at time T in continuous time is...

$$Prob\Big[M_T^- < m, W_T > w\Big] = Prob\Big[W_T < 2m - w\Big] = \int\limits_{\infty}^{2m-w} \frac{1}{\sqrt{2\pi v}}\, e^{-\frac{1}{2v}z^2}\,\delta z \tag{7}$$

Using Equations (4) and (6) above after normalization the equation for the probability as defined by Equation (7) becomes...

$$
\begin{aligned}
P\Bigg[M_T < m, W_T > w\Bigg] &= \int\limits_{\infty}^{2m-w} \frac{1}{\sqrt{2\pi}\sqrt{v}}\, e^{-\frac{1}{2v}z^2}\,\delta z \\
&= \int\limits_{\infty}^{\frac{2m-w}{\sqrt{v}}} \frac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2}x^2}\,\delta x \\
&= N\Bigg[\frac{2m-w}{\sqrt{v}}\Bigg] \qquad (8)
\end{aligned}
$$

Two Hypothetical Problems

Problem 1 - What is the probability that a Brownian motion with zero drift and variance T crosses the barrier 0.75 sometime before time T and ends up below 0.25 at time T assuming that T = 2.00?

The Answer - The answer to our problem using Equation (5) above is...

$$
\begin{aligned}
Prob\Bigg[M_T^{+} > 0.75, W_T < 0.25\Bigg] &= 1 - N\Bigg[\frac{2m-w}{\sqrt{v}}\Bigg] \\
&= 1 - N\Bigg[\frac{(2.00)(0.75)-(0.25)}{\sqrt{2.00}}\Bigg] \\
&= 0.18838 \qquad (9)
\end{aligned}
$$

Problem 2 - What is the probability that a Brownian motion with zero drift and variance equal to $\sigma^2 T$ crosses the barrier -0.25 sometime before time T and ends up above -0.05 at time T assuming that T = 2.00 and $\sigma = 0.80$?

The Answer - The answer to our problem using Equation (8) above is...

$$
\begin{aligned}
Prob\Bigg[M_T^{-} < -0.25, W_T < -0.05\Bigg] &= N\Bigg[\frac{2m-w}{\sqrt{v}}\Bigg] \\
&= N\Bigg[\frac{(2.00)(-0.25)-(-0.05)}{\sqrt{0.80^2 \times 2.00}}\Bigg] \\
&= 0.34541 \qquad (10)
\end{aligned}
$$